UNIVERSIDADE DE BRASILIA
Departamento de Engenharia Mecânica-FT
Prof. Francisco Ricardo da Cunha

SUMÁRIO DOS OPERADORES DIFERENCIAIS, COMPONENTES DO TENSOR TAXA DE DEFORMAÇÃO, VETOR VORTICIDADE E EQUAÇÕES DE BALANÇO EM DIFERENTES SISTEMAS DE COORDENADAS (Retangulares, Cilíndricas e Esféricas)

Aqui

$\underline{v} = \underline{v}(x, y, z, t)$: campo de velocidade

$\rho = \rho(x, y, z, t)$ campo de densidade

$\underline{\underline{\Pi}} \rightarrow$ tensor de tensões em nossa notação $\underline{\underline{T}}$

$\underline{\underline{\Pi}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$

onde $\underline{\underline{\tau}}$ é a parte do tensor de tensões associada a viscosidade

$\dot{\underline{\underline{\gamma}}}$ é o tensor taxa de deformação em nossa notação

$\dot{\underline{\underline{\gamma}}} = 2\underline{\underline{D}}$

TAB 1.

The Equations of Change in Terms of the Fluxes π and q

	In terms of $\dfrac{\partial}{\partial t}$	
Mass:	$\dfrac{\partial}{\partial t}\rho = -(\nabla \cdot \rho v)$	(A)
Momentum:	$\dfrac{\partial}{\partial t}\rho v = -[\nabla \cdot \rho v v] + [\nabla \cdot \pi] + \rho g$	(B)
Energy:	$\dfrac{\partial}{\partial t}\rho \hat{U} = -(\nabla \cdot \rho \hat{U} v) - (\nabla \cdot q) + (\pi : \nabla v)$	(C)
	In terms of $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + (v \cdot \nabla)$	
Mass:	$\dfrac{D\rho}{Dt} = -\rho(\nabla \cdot v)$	(D)
Momentum:	$\rho \dfrac{Dv}{Dt} = [\nabla \cdot \pi] + \rho g$	(E)
Energy:	$\rho \dfrac{D\hat{U}}{Dt} = -(\nabla \cdot q) + (\pi : \nabla v)$	(F)

para fluido Incompressível $\underline{\underline{\tau}} = \mu[\nabla\underline{v} + (\nabla\underline{v})^T]$

$\hat{U}$: energia interna por unidade de massa

$\underline{q}$: fluxo de calor (condução térmica) pela lei de Fourier

$\underline{q} = -\underline{\underline{K}} \cdot \nabla T$

No caso de condução Isotrópica

$\underline{\underline{K}} = k\,\delta_{ij}\hat{e}_i\hat{e}_j = k\underline{\underline{I}}$

$K_{xx} = K_{yy} = K_{zz}$

Equação de Difusão .. suponha que o fluxo de massa $\rho\underline{u}$ é proporcional ao gradiente da densidade ρ. Isto é conhecido como Lei de Fick da difusão. Matematicamente escreve-se que: $\rho\underline{u} = -D\nabla\rho$, onde D é o coeficiente de difusão isotrópica. Esta equação constitutiva é conhecida ser válida empiricamente para pequenos gradientes de densidade. Agora substituindo a lei de Fick na equação da continuidade (A) obtém-se

$$\frac{\partial \rho}{\partial t} = D\,\nabla\cdot(\nabla\rho) = D\nabla^2\rho.$$ Esta é uma equação de difusão.

A equação de difusão acima é geralmente aplicada a difusão de uma espécie através de algum meio contínuo. Neste caso ela é escrita na forma: $\dfrac{\partial c}{\partial t} = D\nabla^2 c$ onde c é a concentração da espécie difusiva.

Aqui

ρ: densidade

μ: viscosidade dinâmica

k: Condutividade Térmica.

TAB 2. **Equations of Change for Newtonian Fluids with Constant ρ, μ, and k**

	Dimensional Forms		Dimensionless Forms[a]	
Continuity	$(\nabla \cdot v) = 0$	(A)	$(\nabla^* \cdot v^*) = 0$	(D)
Motion[b,c]	$\rho \dfrac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g$	(B)	$\mathrm{Re}\dfrac{Dv^*}{Dt^*} = -\nabla^* p^* + \nabla^{*2} v^* + (\mathrm{Re}/\mathrm{Fr})g/g$	(E)
Energy	$\rho \hat{C}_p \dfrac{DT}{Dt} = k\nabla^2 T + \frac{1}{2}\mu(\dot{\gamma} : \dot{\gamma})$	(C)	$\mathrm{P\acute{e}}\dfrac{DT^*}{Dt^*} = \nabla^{*2}T^* + \frac{1}{2}\mathrm{Br}(\dot{\gamma}^* : \dot{\gamma}^*)$	(F)

[a] The dimensionless forms are based on a reference length L, reference velocity V, a reference temperature T_0, and a reference temperature difference ΔT_0. In terms of these $v^* = v/V$, $\nabla^* = L\nabla$, $D/Dt^* = (L/V)D/Dt$, $p^* = (L/\mu V)p$, $T^* = (T - T_0)/\Delta T_0$ and $\dot{\gamma}^* = (L/V)\dot{\gamma}$. The Reynolds number $\mathrm{Re} = LV\rho/\mu$, the Froude number $\mathrm{Fr} = V^2/gL$, the Péclet number $\mathrm{P\acute{e}} = \rho\hat{C}_p LV/k$, and the Brinkman number $\mathrm{Br} = \mu V^2/k\Delta T_0$ are groups that appear as a result of writing the equations in dimensionless form. Other dimensionless groups may enter through the boundary conditions.

[b] For incompressible fluids we may combine the pressure and the gravity terms as $\nabla\mathscr{P} = \nabla p - \rho g$ where $\mathscr{P}$ is called the "modified pressure." If the velocity is specified on the entire boundary, we can conclude that the gravitational acceleration has no effect on the velocity field. If forces are specified on part of the boundary, as in free surface flow, the modified pressure is not a useful concept. The nomenclature "modified pressure" was suggested by G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (1967), p. 176.

[c] The substantial derivative is defined as $D/Dt = \partial/\partial t + (v \cdot \nabla)$.

A.4. Stokes's theorem

Let C be a simple closed curve spanned by a surface S with unit normal $\boldsymbol{n}$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{x} = \int_S (\nabla \wedge \boldsymbol{F}) \cdot \boldsymbol{n}\, dS, \qquad (A.23)$$

where the line integral is taken in an appropriate sense, according to that of $\boldsymbol{n}$ (see Fig. A.1).

Green's theorem in the plane may be viewed as a special case of Stokes's therorem, with $\boldsymbol{F} = [u(x, y), v(x, y), 0]$. If C is a simple closed curve in the x–y plane, and S denotes the region enclosed by C, then

$$\int_C u\, dx + v\, dy = \int_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy. \qquad (A.24)$$

A useful identity derivable from Stokes's theorem is

$$\int_C \phi\, d\boldsymbol{x} = -\int_S (\nabla\phi) \wedge \boldsymbol{n}\, dS. \qquad (A.25)$$

A.5. Orthogonal curvilinear coordinates

Let u, v, and w denote a set of orthogonal curvilinear coordinates, and let $\boldsymbol{e}_u$, $\boldsymbol{e}_v$ and $\boldsymbol{e}_w$ denote unit vectors parallel to the coordinate lines and in the directions of increase of u, v, and w respectively. Then

$$\boldsymbol{e}_u = \boldsymbol{e}_v \wedge \boldsymbol{e}_w, \qquad \text{etc.},$$

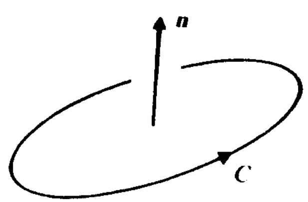

Fig. A.1.

and

$$\delta\boldsymbol{x} = h_1\, \delta u \boldsymbol{e}_u + h_2\, \delta v \boldsymbol{e}_v + h_3\, \delta w \boldsymbol{e}_w,$$

where

$$h_1 = |\partial \boldsymbol{x}/\partial u|, \qquad \text{etc.}$$

Furthermore,

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u}\boldsymbol{e}_u + \frac{1}{h_2}\frac{\partial\phi}{\partial v}\boldsymbol{e}_v + \frac{1}{h_3}\frac{\partial\phi}{\partial w}\boldsymbol{e}_w, \qquad (A.26)$$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 F_u) + \frac{\partial}{\partial v}(h_3 h_1 F_v) + \frac{\partial}{\partial w}(h_1 h_2 F_w)\right]. \qquad (A.27)$$

$$\nabla \wedge \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_u & h_2\boldsymbol{e}_v & h_3\boldsymbol{e}_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1 F_u & h_2 F_v & h_3 F_w \end{vmatrix} \qquad (A.28)$$

For *cylindrical polar coordinates* (Fig. A.2)

$$u = r, \qquad v = \theta, \qquad w = z,$$
$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1.$$

For *spherical polar coordinates* (Fig. A.3)

$$u = r, \qquad v = \theta, \qquad w = \phi,$$
$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r \sin\theta.$$

A.6. Cylindrical polar coordinates

Cylindrical polar coordinates (r, θ, z) are such that

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z,$$

as in Fig. A.2. Clearly,

$$\delta\boldsymbol{x} = \delta r \boldsymbol{e}_r + r\,\delta\theta \boldsymbol{e}_\theta + \delta z \boldsymbol{e}_z$$

and

$$\boldsymbol{e}_r = \cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2, \qquad \boldsymbol{e}_\theta = -\sin\theta\, \boldsymbol{e}_1 + \cos\theta\, \boldsymbol{e}_2, \qquad \boldsymbol{e}_z = \boldsymbol{e}_3.$$

The unit vectors do not change with r or z, but

$$\frac{\partial \boldsymbol{e}_r}{\partial\theta} = \boldsymbol{e}_\theta, \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r, \qquad \frac{\partial \boldsymbol{e}_z}{\partial\theta} = 0. \qquad (A.29)$$

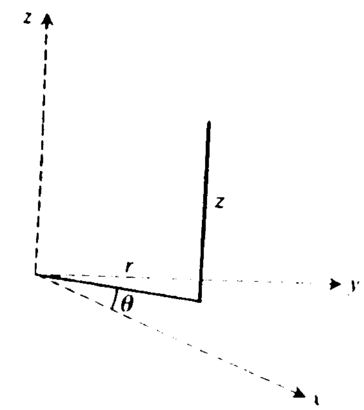

Fig. A.2 Cylindrical polar coordinates.

Also,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial\phi}{\partial z}\boldsymbol{e}_z, \tag{A.30}$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}, \tag{A.31}$$

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}, \tag{A.32}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}, \tag{A.33}$$

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}. \tag{A.34}$$

The Navier–Stokes equations in cylindrical polar coordinates are:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right),$$

$$\tag{A.35}$$

$$\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0.$$

The components of the rate-of-strain tensor are given by:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}, \qquad 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \tag{A.36}$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.$$

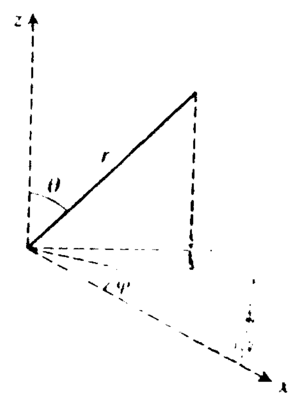

Fig. A.3. Spherical polar coordinates.

A.7. Spherical polar coordinates

Spherical polar coordinates (r, θ, ϕ) are such that

$$x_1 = r\sin\theta\cos\phi, \qquad x_2 = r\sin\theta\sin\phi, \qquad x_3 = r\cos\theta,$$

as in Fig. A.3. Clearly,

$$\delta\boldsymbol{x} = \delta r\,\boldsymbol{e}_r + r\,\delta\theta\,\boldsymbol{e}_\theta + r\sin\theta\,\delta\phi\,\boldsymbol{e}_\phi$$

and

$$\boldsymbol{e}_r = \sin\theta\cos\phi\,\boldsymbol{e}_1 + \sin\theta\sin\phi\,\boldsymbol{e}_2 + \cos\theta\,\boldsymbol{e}_3,$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\,\boldsymbol{e}_1 + \cos\theta\sin\phi\,\boldsymbol{e}_2 - \sin\theta\,\boldsymbol{e}_3,$$

$$\boldsymbol{e}_\phi = -\sin\phi\,\boldsymbol{e}_1 + \cos\phi\,\boldsymbol{e}_2.$$

The unit vectors do not change with r, but

$$\partial\boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta, \qquad \partial\boldsymbol{e}_\theta/\partial\theta = -\boldsymbol{e}_r, \qquad \partial\boldsymbol{e}_\phi/\partial\theta = 0,$$

$$\partial\boldsymbol{e}_r/\partial\phi = \sin\theta\,\boldsymbol{e}_\phi, \qquad \partial\boldsymbol{e}_\theta/\partial\phi = \cos\theta\,\boldsymbol{e}_\phi, \qquad \text{(A.37)}$$

$$\partial\boldsymbol{e}_\phi/\partial\phi = -\sin\theta\,\boldsymbol{e}_r - \cos\theta\,\boldsymbol{e}_\theta.$$

Also,

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\boldsymbol{e}_\phi, \qquad \text{(A.38)}$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}, \qquad \text{(A.39)}$$

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta\,\boldsymbol{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}, \qquad \text{(A40)}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \qquad \text{(A.41)}$$

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad \text{(A.42)}$$

The Navier–Stokes equations in spherical polar coordinates are:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right],$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_ru_\theta}{r} - \frac{u_\phi^2\cot\theta}{r}$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right],$$

$$\frac{\partial u_\phi}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\phi + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi\cot\theta}{r}$$
$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi} + \nu\left[\nabla^2u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r^2\sin^2\theta}\right],$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0. \qquad \text{(A.43)}$$

The components of the rate-of-strain tensor are given by:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r},$$

$$e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$

$$2e_{\theta\phi} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}, \qquad \text{(A.44)}$$

$$2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right),$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.$$

TABELA 1.

Summary of Differential Operations Involving the ∇-Operator in Rectangular Coordinates (x, y, z)

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad \text{(A)}$$

$$(\nabla^2 s) = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \qquad \text{(B)}$$

$$\begin{aligned}(\boldsymbol{\tau} : \nabla \mathbf{v}) = {} & \tau_{xx}\left(\frac{\partial v_x}{\partial x}\right) + \tau_{xy}\left(\frac{\partial v_x}{\partial y}\right) + \tau_{xz}\left(\frac{\partial v_x}{\partial z}\right) \\ & + \tau_{yx}\left(\frac{\partial v_y}{\partial x}\right) + \tau_{yy}\left(\frac{\partial v_y}{\partial y}\right) + \tau_{yz}\left(\frac{\partial v_y}{\partial z}\right) \\ & + \tau_{zx}\left(\frac{\partial v_z}{\partial x}\right) + \tau_{zy}\left(\frac{\partial v_z}{\partial y}\right) + \tau_{zz}\left(\frac{\partial v_z}{\partial z}\right)\end{aligned} \qquad \text{(C)}$$

$$[\nabla s]_x = \frac{\partial s}{\partial x} \qquad \text{(D)}$$

$$[\nabla s]_y = \frac{\partial s}{\partial y} \qquad \text{(E)}$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \qquad \text{(F)}$$

$$[\nabla \times \mathbf{v}]_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \qquad \text{(G)}$$

$$[\nabla \times \mathbf{v}]_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \qquad \text{(H)}$$

$$[\nabla \times \mathbf{v}]_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \qquad \text{(I)}$$

$$[\nabla \cdot \boldsymbol{\tau}]_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \qquad \text{(J)}$$

$$[\nabla \cdot \boldsymbol{\tau}]_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \qquad \text{(K)}$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \qquad \text{(L)}$$

$$[\nabla^2 \mathbf{v}]_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \qquad \text{(M)}$$

$$[\nabla^2 \mathbf{v}]_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \qquad \text{(N)}$$

$$[\nabla^2 \mathbf{v}]_z = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \qquad \text{(O)}$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_x = v_x\left(\frac{\partial w_x}{\partial x}\right) + v_y\left(\frac{\partial w_x}{\partial y}\right) + v_z\left(\frac{\partial w_x}{\partial z}\right) \qquad \text{(P)}$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_y = v_x\left(\frac{\partial w_y}{\partial x}\right) + v_y\left(\frac{\partial w_y}{\partial y}\right) + v_z\left(\frac{\partial w_y}{\partial z}\right) \qquad \text{(Q)}$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_z = v_x\left(\frac{\partial w_z}{\partial x}\right) + v_y\left(\frac{\partial w_z}{\partial y}\right) + v_z\left(\frac{\partial w_z}{\partial z}\right) \qquad \text{(R)}$$

Continuação TAB 1

$$\{\nabla \boldsymbol{v}\}_{xx} = \frac{\partial v_x}{\partial x} \qquad \text{(S)}$$

$$\{\nabla \boldsymbol{v}\}_{xy} = \frac{\partial v_y}{\partial x} \qquad \text{(T)}$$

$$\{\nabla \boldsymbol{v}\}_{xz} = \frac{\partial v_z}{\partial x} \qquad \text{(U)}$$

$$\{\nabla \boldsymbol{v}\}_{yx} = \frac{\partial v_x}{\partial y} \qquad \text{(V)}$$

$$\{\nabla \boldsymbol{v}\}_{yy} = \frac{\partial v_y}{\partial y} \qquad \text{(W)}$$

$$\{\nabla \boldsymbol{v}\}_{yz} = \frac{\partial v_z}{\partial y} \qquad \text{(X)}$$

$$\{\nabla \boldsymbol{v}\}_{zx} = \frac{\partial v_x}{\partial z} \qquad \text{(Y)}$$

$$\{\nabla \boldsymbol{v}\}_{zy} = \frac{\partial v_y}{\partial z} \qquad \text{(Z)}$$

$$\{\nabla \boldsymbol{v}\}_{zz} = \frac{\partial v_z}{\partial z} \qquad \text{(AA)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{xx} = (\boldsymbol{v} \cdot \nabla)\tau_{xx} \qquad \text{(BB)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{xy} = (\boldsymbol{v} \cdot \nabla)\tau_{xy} \qquad \text{(CC)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{xz} = (\boldsymbol{v} \cdot \nabla)\tau_{xz} \qquad \text{(DD)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{yx} = (\boldsymbol{v} \cdot \nabla)\tau_{yx} \qquad \text{(EE)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{yy} = (\boldsymbol{v} \cdot \nabla)\tau_{yy} \qquad \text{(FF)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{yz} = (\boldsymbol{v} \cdot \nabla)\tau_{yz} \qquad \text{(GG)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{zx} = (\boldsymbol{v} \cdot \nabla)\tau_{zx} \qquad \text{(HH)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{zy} = (\boldsymbol{v} \cdot \nabla)\tau_{zy} \qquad \text{(II)}$$

$$\{\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}\}_{zz} = (\boldsymbol{v} \cdot \nabla)\tau_{zz} \qquad \text{(JJ)}$$

where the operator $(\boldsymbol{v} \cdot \nabla) = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$

TABELA 2

Summary of Differential Operations Involving the ∇-Operator in Cylindrical Coordinates (r, θ, z)

$$(\nabla \cdot v) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \quad \text{(A)}$$

$$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \quad \text{(B)}$$

$$\begin{aligned}(\tau : \nabla v) = {} & \tau_{rr}\left(\frac{\partial v_r}{\partial r}\right) + \tau_{r\theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \tau_{rz}\left(\frac{\partial v_r}{\partial z}\right) \\ & + \tau_{\theta r}\left(\frac{\partial v_\theta}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \tau_{\theta z}\left(\frac{\partial v_\theta}{\partial z}\right) \\ & + \tau_{zr}\left(\frac{\partial v_z}{\partial r}\right) + \tau_{z\theta}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) + \tau_{zz}\left(\frac{\partial v_z}{\partial z}\right)\end{aligned} \quad \text{(C)}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r} \quad \text{(D)} \qquad [\nabla \times v]_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad \text{(G)}$$

$$[\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta} \quad \text{(E)} \qquad [\nabla \times v]_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \quad \text{(H)}$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \quad \text{(F)} \qquad [\nabla \times v]_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \quad \text{(I)}$$

$$[\nabla \cdot \tau]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r} \quad \text{(J)}$$

$$[\nabla \cdot \tau]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r} \quad \text{(K)}$$

$$[\nabla \cdot \tau]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz} \quad \text{(L)}$$

$$[\nabla^2 v]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \quad \text{(M)}$$

$$[\nabla^2 v]_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \quad \text{(N)}$$

$$[\nabla^2 v]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \quad \text{(O)}$$

$$[v \cdot \nabla w]_r = v_r\left(\frac{\partial w_r}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + v_z\left(\frac{\partial w_r}{\partial z}\right) \quad \text{(P)}$$

$$[v \cdot \nabla w]_\theta = v_r\left(\frac{\partial w_\theta}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + v_z\left(\frac{\partial w_\theta}{\partial z}\right) \quad \text{(Q)}$$

$$[v \cdot \nabla w]_z = v_r\left(\frac{\partial w_z}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_z}{\partial \theta}\right) + v_z\left(\frac{\partial w_z}{\partial z}\right) \quad \text{(R)}$$

Continuação TAB 2

$\{\nabla \mathbf{v}\}_{rr} = \dfrac{\partial v_r}{\partial r}$	(S)
$\{\nabla \mathbf{v}\}_{r\theta} = \dfrac{\partial v_\theta}{\partial r}$	(T)
$\{\nabla \mathbf{v}\}_{rz} = \dfrac{\partial v_z}{\partial r}$	(U)
$\{\nabla \mathbf{v}\}_{\theta r} = \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta}{r}$	(V)
$\{\nabla \mathbf{v}\}_{\theta\theta} = \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r}{r}$	(W)
$\{\nabla \mathbf{v}\}_{\theta z} = \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta}$	(X)
$\{\nabla \mathbf{v}\}_{zr} = \dfrac{\partial v_r}{\partial z}$	(Y)
$\{\nabla \mathbf{v}\}_{z\theta} = \dfrac{\partial v_\theta}{\partial z}$	(Z)
$\{\nabla \mathbf{v}\}_{zz} = \dfrac{\partial v_z}{\partial z}$	(AA)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{rr} = (\mathbf{v}\cdot\nabla)\tau_{rr} - \dfrac{v_\theta}{r}(\tau_{r\theta} + \tau_{\theta r})$	(BB)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{r\theta} = (\mathbf{v}\cdot\nabla)\tau_{r\theta} + \dfrac{v_\theta}{r}(\tau_{rr} - \tau_{\theta\theta})$	(CC)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{rz} = (\mathbf{v}\cdot\nabla)\tau_{rz} - \dfrac{v_\theta}{r}\tau_{\theta z}$	(DD)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta r} = (\mathbf{v}\cdot\nabla)\tau_{\theta r} + \dfrac{v_\theta}{r}(\tau_{rr} - \tau_{\theta\theta})$	(EE)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta\theta} = (\mathbf{v}\cdot\nabla)\tau_{\theta\theta} + \dfrac{v_\theta}{r}(\tau_{r\theta} + \tau_{\theta r})$	(FF)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{\theta z} = (\mathbf{v}\cdot\nabla)\tau_{\theta z} + \dfrac{v_\theta}{r}\tau_{rz}$	(GG)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{zr} = (\mathbf{v}\cdot\nabla)\tau_{zr} - \dfrac{v_\theta}{r}\tau_{z\theta}$	(HH)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{z\theta} = (\mathbf{v}\cdot\nabla)\tau_{z\theta} + \dfrac{v_\theta}{r}\tau_{zr}$	(II)
$\{\mathbf{v}\cdot\nabla\boldsymbol{\tau}\}_{zz} = (\mathbf{v}\cdot\nabla)\tau_{zz}$	(JJ)

where the operator $(\mathbf{v}\cdot\nabla) = v_r\dfrac{\partial}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial}{\partial\theta} + v_z\dfrac{\partial}{\partial z}$

TABELA 3

Summary of Differential Operations Involving the ∇-Operator in Spherical Coordinates (r, θ, ϕ)

$$(\nabla\cdot\mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \qquad \text{(A)}$$

$$(\nabla^2 s) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial s}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 s}{\partial\phi^2} \qquad \text{(B)}$$

$$\begin{aligned}(\boldsymbol{\tau}:\nabla\mathbf{v}) = {} & \tau_{rr}\left(\frac{\partial v_r}{\partial r}\right) + \tau_{r\theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}\right) + \tau_{r\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\phi}{r}\right)\\ & + \tau_{\theta r}\left(\frac{\partial v_\theta}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right) + \tau_{\theta\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v_\phi}{r}\cot\theta\right)\\ & + \tau_{\phi r}\left(\frac{\partial v_\phi}{\partial r}\right) + \tau_{\phi\theta}\left(\frac{1}{r}\frac{\partial v_\phi}{\partial\theta}\right) + \tau_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta\right)\end{aligned} \qquad \text{(C)}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r} \qquad \text{(D)}$$

$$[\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial\theta} \qquad \text{(E)}$$

$$[\nabla s]_\phi = \frac{1}{r\sin\theta}\frac{\partial s}{\partial\phi} \qquad \text{(F)}$$

$$[\nabla\times\mathbf{v}]_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} \qquad \text{(G)}$$

$$[\nabla\times\mathbf{v}]_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi) \qquad \text{(H)}$$

$$[\nabla\times\mathbf{v}]_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta} \qquad \text{(I)}$$

$$[\nabla\cdot\boldsymbol{\tau}]_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta r}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tau_{\phi r} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r} \qquad \text{(J)}$$

$$[\nabla\cdot\boldsymbol{\tau}]_\theta = \frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tau_{\phi\theta} + \frac{(\tau_{\theta r}-\tau_{r\theta}) - \tau_{\phi\phi}\cot\theta}{r} \qquad \text{(K)}$$

$$[\nabla\cdot\boldsymbol{\tau}]_\phi = \frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\phi}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tau_{\phi\phi} + \frac{(\tau_{\phi r}-\tau_{r\phi}) + \tau_{\phi\theta}\cot\theta}{r} \qquad \text{(L)}$$

$$\begin{aligned}[\nabla^2\mathbf{v}]_r = {} & \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\\ & - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\end{aligned} \qquad \text{(M)}$$

$$[\nabla^2\mathbf{v}]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi} \qquad \text{(N)}$$

$$\begin{aligned}[\nabla^2\mathbf{v}]_\phi = {} & \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}\\ & + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}\end{aligned} \qquad \text{(O)}$$

$$[\mathbf{v}\cdot\nabla\mathbf{w}]_r = v_r\left(\frac{\partial w_r}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial\theta} - \frac{w_\theta}{r}\right) + v_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_r}{\partial\phi} - \frac{w_\phi}{r}\right) \qquad \text{(P)}$$

$$[\mathbf{v}\cdot\nabla\mathbf{w}]_\theta = v_r\left(\frac{\partial w_\theta}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial\theta} + \frac{w_r}{r}\right) + v_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_\theta}{\partial\phi} - \frac{w_\phi}{r}\cot\theta\right) \qquad \text{(Q)}$$

Continuação TAB 3.

$$[\mathbf{v}\cdot\nabla\mathbf{w}]_\phi = v_r\left(\frac{\partial w_\phi}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\phi}{\partial \theta}\right) + v_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_\phi}{\partial \phi} + \frac{w_r}{r} + \frac{w_\theta}{r}\cot\theta\right) \quad \text{(R)}$$

$$\{\nabla\mathbf{v}\}_{rr} = \frac{\partial v_r}{\partial r} \quad \text{(S)}$$

$$\{\nabla\mathbf{v}\}_{r\theta} = \frac{\partial v_\theta}{\partial r} \quad \text{(T)}$$

$$\{\nabla\mathbf{v}\}_{r\phi} = \frac{\partial v_\phi}{\partial r} \quad \text{(U)}$$

$$\{\nabla\mathbf{v}\}_{\theta r} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \quad \text{(V)}$$

$$\{\nabla\mathbf{v}\}_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \quad \text{(W)}$$

$$\{\nabla\mathbf{v}\}_{\theta\phi} = \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} \quad \text{(X)}$$

$$\{\nabla\mathbf{v}\}_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} \quad \text{(Y)}$$

$$\{\nabla\mathbf{v}\}_{\phi\theta} = \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r}\cot\theta \quad \text{(Z)}$$

$$\{\nabla\mathbf{v}\}_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta \quad \text{(AA)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{rr} = (\mathbf{v}\cdot\nabla)\tau_{rr} - \left(\frac{v_\theta}{r}\right)(\tau_{r\theta} + \tau_{\theta r}) - \left(\frac{v_\phi}{r}\right)(\tau_{r\phi} + \tau_{\phi r}) \quad \text{(BB)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{r\theta} = (\mathbf{v}\cdot\nabla)\tau_{r\theta} + \left(\frac{v_\theta}{r}\right)(\tau_{rr} - \tau_{\theta\theta}) - \left(\frac{v_\phi}{r}\right)(\tau_{\phi\theta} + \tau_{r\phi}\cot\theta) \quad \text{(CC)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{r\phi} = (\mathbf{v}\cdot\nabla)\tau_{r\phi} - \left(\frac{v_\theta}{r}\right)\tau_{\theta\phi} + \left(\frac{v_\phi}{r}\right)[(\tau_{rr} - \tau_{\phi\phi}) + \tau_{r\theta}\cot\theta] \quad \text{(DD)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{\theta r} = (\mathbf{v}\cdot\nabla)\tau_{\theta r} + \left(\frac{v_\theta}{r}\right)(\tau_{rr} - \tau_{\theta\theta}) - \left(\frac{v_\phi}{r}\right)(\tau_{\theta\phi} + \tau_{\phi r}\cot\theta) \quad \text{(EE)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{\theta\theta} = (\mathbf{v}\cdot\nabla)\tau_{\theta\theta} + \left(\frac{v_\theta}{r}\right)(\tau_{r\theta} + \tau_{\theta r}) - \left(\frac{v_\phi}{r}\right)(\tau_{\theta\phi} + \tau_{\phi\theta})\cot\theta \quad \text{(FF)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{\theta\phi} = (\mathbf{v}\cdot\nabla)\tau_{\theta\phi} + \left(\frac{v_\theta}{r}\right)\tau_{r\phi} + \left(\frac{v_\phi}{r}\right)[\tau_{\theta r} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta] \quad \text{(GG)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{\phi r} = (\mathbf{v}\cdot\nabla)\tau_{\phi r} - \left(\frac{v_\theta}{r}\right)\tau_{\phi\theta} + \left(\frac{v_\phi}{r}\right)[(\tau_{rr} - \tau_{\phi\phi}) + \tau_{\theta r}\cot\theta] \quad \text{(HH)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{\phi\theta} = (\mathbf{v}\cdot\nabla)\tau_{\phi\theta} + \left(\frac{v_\theta}{r}\right)\tau_{\phi r} + \left(\frac{v_\phi}{r}\right)[\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta] \quad \text{(II)}$$

$$\{\mathbf{v}\cdot\nabla\tau\}_{\phi\phi} = (\mathbf{v}\cdot\nabla)\tau_{\phi\phi} + \left(\frac{v_\phi}{r}\right)[(\tau_{r\phi} + \tau_{\phi r}) + (\tau_{\theta\phi} + \tau_{\phi\theta})\cot\theta] \quad \text{(JJ)}$$

where the operator $(\mathbf{v}\cdot\nabla) = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$

TABELA 4

Rectangular Coordinates (x, y, z):

$$\dot{\gamma}_{xx} = 2\frac{\partial v_x}{\partial x}$$

$$\dot{\gamma}_{yy} = 2\frac{\partial v_y}{\partial y}$$

$$\dot{\gamma}_{zz} = 2\frac{\partial v_z}{\partial z}$$

$$\dot{\gamma}_{xy} = \dot{\gamma}_{yx} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}$$

$$\dot{\gamma}_{yz} = \dot{\gamma}_{zy} = \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}$$

$$\dot{\gamma}_{zx} = \dot{\gamma}_{xz} = \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}$$

Cylindrical Coordinates (r, θ, z):

$$\dot{\gamma}_{rr} = 2\frac{\partial v_r}{\partial r}$$

$$\dot{\gamma}_{\theta\theta} = 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\dot{\gamma}_{zz} = 2\frac{\partial v_z}{\partial z}$$

$$\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

$$\dot{\gamma}_{\theta z} = \dot{\gamma}_{z\theta} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}$$

$$\dot{\gamma}_{zr} = \dot{\gamma}_{rz} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}$$

Spherical Coordinates (r, θ, ϕ):

$$\dot{\gamma}_{rr} = 2\frac{\partial v_r}{\partial r}$$

$$\dot{\gamma}_{\theta\theta} = 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\dot{\gamma}_{\phi\phi} = 2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)$$

• COMPONENTES DO TENSOR TAXA DE DEFORMAÇÃO

$$\underline{\underline{\dot{\gamma}}} = (\nabla \underline{v} + (\nabla \underline{v})^t)$$

* Note que existe uma diferença da definição que foi apresentada em aula apenas devido a constante $(1/2)$

$$\begin{cases} \underline{\underline{D}} = \underline{\underline{\dot{\gamma}}} = \frac{1}{2}(\nabla \underline{v} + (\nabla \underline{v})^t) \\ \underline{\underline{W}} = \frac{1}{2}(\nabla \underline{v} - (\nabla \underline{v})^t) \end{cases}$$

Assim todos os resultados das tabelas 4 e 5 devem ser multiplicados por $(1/2)$ p/ manter coerência da definição:

$$\nabla \underline{v} = \underline{\underline{D}} + \underline{\underline{W}}.$$

CONTINUAÇÃO DA TABELA 4.

$$\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} \quad \text{(B.3-16)}$$

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_{\phi\theta} = \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \quad \text{(B.3-17)}$$

$$\dot{\gamma}_{\phi r} = \dot{\gamma}_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \quad \text{(B.3-18)}$$

TABELA 5

● COMPONENTES DO TENSOR VORTICIDADE

$$\underline{\underline{W}} = (\nabla \underline{v} - (\nabla \underline{v})^t)$$

Rectangular Coordinates (x, y, z):

$$\omega_{xy} = -\omega_{yx} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad \text{(B.4-1)}$$

$$\omega_{yz} = -\omega_{zy} = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \quad \text{(B.4-2)}$$

$$\omega_{zx} = -\omega_{xz} = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \quad \text{(B.4-3)}$$

Cylindrical Coordinates (r, θ, z):

$$\omega_{r\theta} = -\omega_{\theta r} = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \quad \text{(B.4-4)}$$

$$\omega_{\theta z} = -\omega_{z\theta} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad \text{(B.4-5)}$$

$$\omega_{zr} = -\omega_{rz} = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \quad \text{(B.4-6)}$$

Spherical Coordinates (r, θ, ϕ):

$$\omega_{r\theta} = -\omega_{\theta r} = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} \quad \text{(B.4-7)}$$

$$\omega_{\theta\phi} = -\omega_{\phi\theta} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \quad \text{(B.4-8)}$$

$$\omega_{\phi r} = -\omega_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi) \quad \text{(B.4-9)}$$